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The quantum double and related constructions

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Abstract

This paper describes various constructions, on a given bialgebra B, producing bialgebras with extra structure. The monoidal categories of representations of the constructed bialgebras are described in terms of the monoidal category of representations of B. Some of these results were known only for a finite-dimensional Hopf algebra B. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

The quantum double of a finite-dimensional Hopf algebra was defined by Drinfeld [6]. The centre of a monoidal category was defined in [10] and used to show that the category of tangles on ribbons has two universal properties. The close relationship between the double and centre constructions, and the relationship to Yetter's crossed bimodules [24], was clarified by Majid [17]; see accounts in [13] and [20].

The purpose of the present paper is to provide a more detailed analysis of this correspondence between constructions on bialgebras and constructions on monoidal categories. Many of the ideas herein have their roots in [10, 11, 14], but we now show that the double of a monoidal category, as defined by [14], can be seen as the composite of three monoidal constructions, all used in [10] for the specific purpose there, yet worthy of extra study. Each of these constructions has its own universal property.

By making use of the representation theorem [11, Section 7] underlying Tannaka duality, this paper provides a fully general construction of the quantum double $D(B)$ of any (perhaps infinite dimensional) bialgebra B and describes the category of representaions of $D(B)$ in terms of the category of representations of B. The papers [1, 18]

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provide explicit constructions of the double of a quantum group which is compact and locally compact, respectively; however, other authors [6, 13, p. 213, p. 333; 14, 20, p. 189] restrict attention to the finite-dimensional case.

The definition of monoidal (= tensor), braided monoidal, balanced monoidal and tortile (= ribbon) monoidal category can be found in each of the works [11–13, 21], and elsewhere. The author's view is that an abstract quantum group is a cotortile Hopf algebra [11, p. 447] and that a representation is a finite-dimensional comodule. So the quantum double of H should be a quantum group and the category of representations should be related to the category of representations of H. Other authors have worked in terms of tortile Hopf algebras and their modules; this makes essentially no difference when the Hopf algebras are finite dimensional.

1. Creating a braiding

A description of the centre of a monoidal category from the viewpoint of higher-dimensional categories will be given. As this centre is not an adjoint to an obvious functor, it seems necessary to explain why one might predict such a construction. However, as no use is made of this approach in the rest of the paper, readers not familiar with higher categories should move on to the explicit description given soon after. Readers conversant with bicategories and their homomorphisms (in the sense of Bénabou [3]) will be able to deduce the equivalence of the two constructions.

In any monoidal category $\mathscr{M}$, the endomorphisms of the unit I for tensor product form a *commutative* monoid End(I) whose operation is composition (which turns out to be the same as the operation of tensoring endomorphisms by the so-called Eckmann–Hilton argument). By regarding a monoid M as a one-object category ΣM and taking $\mathscr{M}$ to be the strict monoidal endofunctor category of ΣM with composition as tensor product, clearly the unit I for tensor product is the identity functor of ΣM and End(I) is the *centre* $\mathscr{Z}(M)$ of M.

Consider the corresponding situation after raising the dimension by 1 in moving from categories to bicategories. The definition of monoidal bicategory $\mathscr{M}$ appears in many places [2, 4, 5, 8, 15]. Proposition 5.3 of [12] implies that the category End(I) of endomorphisms of the unit I for tensor product is braided monoidal. By regarding a monoidal category $\mathscr{V}$ as a one-object bicategory $\Sigma\mathscr{V}$, we can look at the monoidal bicategory Hom($\Sigma\mathscr{V}, \Sigma\mathscr{V}$) of endo-homomorphisms of $\Sigma\mathscr{V}$ where the tensor product is composition. The braided monoidal category Hom$(\Sigma\mathscr{V}, \Sigma\mathscr{V})(1_{\mathscr{V}}, 1_{\mathscr{V}})$ is called the centre of $\mathscr{V}$: an explicit description will be provided below. Even higher-dimensional versions of centre have now been envisaged by Baez–Neuchl [2].

Let $\mathscr{V}$ be any monoidal category (although we write as if it were strict). The *centre* $\mathscr{Z}(\mathscr{V})$ of $\mathscr{V}$ is the braided monoidal category described as follows. The objects are pairs (A, u) consisting of an object A of $\mathscr{V}$ and a family u of invertible arrows

$u_x: A\otimes X \to X\otimes A$ in $\mathscr{V}$ which are natural in X and satisfy the following commutativity condition:

$$\begin{array}{ccc} A\otimes X\otimes Y & \xrightarrow{\quad u_{X\otimes Y}\quad} & X\otimes Y\otimes A \\ & {\scriptstyle u_X\otimes 1}\searrow \quad \nearrow{\scriptstyle 1\otimes u_Y} & \\ & X\otimes A\otimes Y & \end{array}$$

An arrow $f: (A, u) \to (B, v)$ in $\mathscr{Z}(\mathscr{V})$ is an arrow $f: A \to B$ in $\mathscr{V}$ such that the following square commutes for all objects X of $\mathscr{V}$:

$$\begin{array}{ccc} A\otimes X & \xrightarrow{u_X} & X\otimes A \\ {\scriptstyle f\otimes 1}\downarrow & & \downarrow{\scriptstyle 1\otimes f} \\ B\otimes X & \xrightarrow{v_X} & X\otimes B \end{array}$$

The tensor product is given by $(A, u)\otimes(B, v) = (A\otimes B, w)$, where

$$w_x = (u_x\otimes 1)\circ(1\otimes v_x).$$

The braiding $c_{(A,u),(B,v)}: (A, u)\otimes(B, v) \to (B, v)\otimes(A, u)$ is the arrow

$$u_B: A\otimes B \to B\otimes A.$$

There is a (strict) monoidal forgetful functor $\mathscr{Z}(\mathscr{V}) \to \mathscr{V}$ taking (A, u) to A.

Let us define a monoidal category $\mathscr{V}$ to be *abelian* when it is an abelian category [16, p. 194] and, for each object X, the functors $X\otimes$ - and - $\otimes X$ are additive and exact.

Proposition 1. *If $\mathscr{V}$ is an abelian monoidal category then so is the centre $\mathscr{Z}(\mathscr{V})$ of $\mathscr{V}$ and, moreover, the forgetful functor $\mathscr{Z}(\mathscr{V}) \to \mathscr{V}$ is exact.*

Proof. The zero object 0 of $\mathscr{V}$ obviously lifts to an object of the centre since $X\otimes 0$ and $0\otimes X$ are zero objects; this gives a zero object for $\mathscr{Z}(\mathscr{V})$. For objects (A, u), (B, v) of the centre, we have $(A, u)\oplus(B, v) = (A\oplus B, w)$ where $w_x: (A\oplus B)\otimes X \to X\otimes(A\oplus B)$ is obtained by transporting $u_x\oplus v_x$ across the canonical isomorphisms

$$(A\oplus B)\otimes X \cong (A\otimes X)\oplus(B\otimes X),\ X\otimes(A\oplus B) \cong (X\otimes A)\oplus(X\otimes B).$$

For any arrow $f: (A, u) \to (B, v)$ in $\mathscr{Z}(\mathscr{V})$, let $k: K \to A$, $q: A \to C$ be the kernel, cokernel of f in $\mathscr{V}$. Since the rows of the following diagram are exact, there are unique dashed arrows making the diagram commute:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & K\otimes X & \xrightarrow{k\otimes 1} & A\otimes X & \xrightarrow{f\otimes 1} & B\otimes X & \xrightarrow{q\otimes 1} & C\otimes X & \longrightarrow & 0 \\ & & {\scriptstyle r_X}\vdots & & {\scriptstyle u_X}\downarrow & & \downarrow{\scriptstyle v_X} & & \vdots{\scriptstyle w_X} & & \\ 0 & \longrightarrow & X\otimes K & \xrightarrow{1\otimes k} & X\otimes A & \xrightarrow{1\otimes f} & X\otimes B & \xrightarrow{1\otimes q} & X\otimes C & \longrightarrow & 0 \end{array}$$

It is easy to see that $k: (K, r) \to (A, u)$, $q: (B, v) \to (C, w)$ are the kernel, cokernel of $f: (A, u) \to (B, v)$ in $\mathscr{Z}(\mathscr{V})$. So every arrow in $\mathscr{Z}(\mathscr{V})$ has a kernel and cokernel. An arrow is a monomorphism, epimorphism iff its kernel, cokernel is zero; so $f: (A, u) \to (B, v)$ in $\mathscr{Z}(\mathscr{V})$ is a monomorphism, epimorphism iff $f: A \to B$ is so in $\mathscr{V}$. That every monomorphism is the kernel of its cokernel (and dually for epimorphisms) now follows from that property in $\mathscr{V}$. The last clause of the proposition is clear from the constructions. □

We recall some terminology from [11, Sections 7–10]. For any coalgebra B over the complex numbers (for simplicity), we write $Comod_f(B)$ for the category of B-comodules which are finite dimensional as vector spaces. If B is a bialgebra, there is a classical monoidal structure on this category such that the forgetful functor $Comod_f(B) \to Vect$ is strong monoidal. The cobraidings on B are in bijection with the braidings on $Comod_f(B)$.

Proposition 2. *Suppose B is a bialgebra. Then there exists a cobraided bialgebra $D_1(B)$ unique up to isomorphism with the property that there is an equivalence of braided monoidal categories*

$$Comod_f(D_1(B)) \xrightarrow{\sim} \mathscr{Z}(Comod_f(B)),$$

which commutes with the forgetful functors into Vect.

Proof. We apply the representation theorem [11, Section 7, p. 456] to the composite UP of the underlying functors

$$P: \mathscr{Z}(Comod_f(B)) \longrightarrow Comod_f(B), \qquad U: Comod_f(B) \longrightarrow \mathscr{V}ect;$$

the hypotheses apply to UP by Proposition 1. So we obtain a coalgebra

$$D_1(B) = \mathrm{End}^{\vee}(UP)$$

and an equivalence of categories $Comod_f(D_1(B)) \xrightarrow{\sim} \mathscr{Z}(Comod_f(B))$ commuting with the forgetful functors into *Vect*. The monoidal structure on $\mathscr{Z}(Comod_f(B))$ which is preserved by UP yields a bialgebra structure on $D_1(B)$ enriching the equivalence of categories to a strong monoidal one; see [11, Section 8], for example. Moreover, the braiding on $\mathscr{Z}(Comod_f(B))$ yields a cobraiding on $D_1(B)$ enriching the equivalence to a braided one [11, Section 10, Proposition 3 and Section 7, Proposition 1]. □

Proposition 3. *Suppose $\mathscr{V}$ is any monoidal category. An object (A, u) of the centre $\mathscr{Z}(\mathscr{V})$ has a left dual if and only if the object A of $\mathscr{V}$ has a left dual A^* and the mate $v: A^* \otimes X \to X \otimes A^*$ of $u_X^{-1}: X \otimes A \to A \otimes X$ is invertible. In this case, the left dual of (A, u) is (A^*, v).*

Proof. Suppose (A, u) has a left dual (A^*, v) in the centre with counit $\varepsilon : (A^*, v) \otimes (A, u) \to I$. Then A^* is a left dual for A in $\mathscr{V}$ with counit ε since the forgetful functor is strong monoidal and so preserves duals. The condition that ε should be an arrow of the centre can be expressed as commutativity of the following square which also expresses the condition that v_X should be the mate of u_X^{-1}; and, of course, v_X is invertible:

$$\begin{array}{ccc} A^* \otimes X \otimes A & \xrightarrow{v_X \otimes 1} & X \otimes A^* \otimes A \\ {\scriptstyle 1 \otimes u_X^{-1}} \downarrow & & \downarrow {\scriptstyle 1 \otimes \varepsilon} \\ A^* \otimes A \otimes X & \xrightarrow{\quad \varepsilon \otimes 1 \quad} & X \end{array}$$

Conversely, suppose the dual A^* of A exists and the mate v_X of u_X^{-1} is invertible. By the usual "mate calculus" we see that (A^*, v) is indeed an object of the centre. Since v_X is defined by the above commutative square, it follows that $\varepsilon : (A^*, v) \otimes (A, u) \to I$ is an arrow of the centre. A similar argument applies to give the unit as an arrow $\eta : I \to (A, u) \otimes (A^*, v)$ in the centre. It follows that (A^*, v) is left dual to (A, u) since the equations required of ε, η for an adjunction are the same in $\mathscr{Z}(\mathscr{V})$ as in $\mathscr{V}$. □

The centre construction has a universal property. It does *not* provide the right biadjoint [22] of the forgetful 2-functor from braided monoidal categories to monoidal categories, but it does satisfy a restricted version of that property. Let us define a functor $S : \mathscr{C} \to \mathscr{V}$ to be *cauchy dense* when every object of $\mathscr{V}$ is a retract of one of the form $S(U)$ for some object U of $\mathscr{C}$. If $\mathscr{C}$ is braided, it is fairly easy to see that every full, cauchy dense, strong monoidal functor $S : \mathscr{C} \to \mathscr{V}$ can be lifted, uniquely up to isomorphism, to a braided strong monoidal functor $T : \mathscr{C} \to \mathscr{Z}(\mathscr{V})$; compare [10, Proposition 4, p. 47] and [13, Proposition XIII.4.3, p. 332].

For completeness, we should mention that the forgetful 2-functor from the 2-category of braided monoidal categories (with braided strong monoidal functors as the arrows) to the 2-category of monoidal categories (with strong monoidal functors as the arrows) does have a left biadjoint. This follows from a standard "adjoint triangle" argument using the free monoidal and free braided monoidal category constructions on a category [12, Sections 1, 2]. To be explicit, for each monoidal category $\mathscr{V}$, there is a braided monoidal category $\mathscr{br}(\mathscr{V})$, such that for all braided monoidal categories $\mathscr{C}$, the category of braided strong monoidal functors $\mathscr{br}(\mathscr{V}) \to \mathscr{C}$ (with monoidal natural transformations as the arrows) is (pseudonaturally) equivalent to the category of strong monoidal functors $\mathscr{V} \to \mathscr{C}$ (with monoidal natural transformations as the arrows).

2. Creating a twist

For any category $\mathscr{V}$, there is the category $\mathscr{V}^{\mathbf{Z}}$ of automorphisms in $\mathscr{V}$. The objects are pairs (A, ξ) where $\xi : A \to A$ is an automorphism of A in $\mathscr{V}$. An arrow

$f: (A, \zeta) \to (B, \zeta)$ in $\mathscr{V}^Z$ is an arrow $f: A \to B$ in $\mathscr{V}$ such that $f\xi = \zeta f$. If $\mathscr{V}$ is a braided monoidal category, there is a monoidal structure on $\mathscr{V}^Z$ given by $(A, \xi) \otimes (B, \zeta) = (A \otimes B, c_{B,A} c_{A,B}(\xi \otimes \zeta))$ (see [10, p. 49]). Indeed, the monoidal category $\mathscr{V}^Z$ is balanced. The braiding is uniquely determined by the condition that the forgetful functor $\mathscr{V}^Z \to \mathscr{V}$ is braided; the naturality of the braiding in $\mathscr{V}$ shows that it commutes with the appropriate automorphisms. The twist

$$\theta_{(A,\xi)}: (A, \xi) \to (A, \xi),$$

can be taken to be ξ itself; the following calculation proves the twist condition:

$$\begin{aligned}\theta_{(A,\xi)\otimes(B,\zeta)} &= c_{B,A} c_{A,B}(\xi \otimes \zeta)\\ &= c_{B,A}(\zeta \otimes \xi) c_{A,B}\\ &= c_{(B,\zeta),(A,\xi)}(\theta_{(B,\zeta)} \otimes \theta_{(A,\xi)}) c_{(A,\xi),(B,\zeta)}.\end{aligned}$$

Proposition 4. *If $\mathscr{V}$ is a braided abelian monoidal category then the category $\mathscr{V}^Z$ of automorphisms in $\mathscr{V}$ is balanced abelian monoidal and, moreover, the forgetful functor $\mathscr{V}^Z \to \mathscr{V}$ is exact.*

Proof. Every category of functors from a given category into an abelian category is abelian and the evaluation functors are left exact. So the category $\mathscr{V}^Z$ is abelian and the functor $\mathscr{V}^Z \to \mathscr{V}$ is exact. Since the forgetful functor $\mathscr{V}^Z \to \mathscr{V}$ is strong monoidal, conservative and exact, the monoidal category $\mathscr{V}^Z$ is abelian. □

Proposition 5. *Suppose B is a cobraided bialgebra. Then there exists a cobalanced bialgebra $D_2(B)$ unique up to isomorphism with the property that there is an equivalence of balanced monoidal categories*

$$Comod_f(D_2(B)) \xrightarrow{\sim} Comod_f(B)^Z,$$

which commutes with the forgetful functors into Vect.

Proof. Proceed along the lines of the proof of Proposition 2. □

In order to discuss duality in the automorphism category of a braided monoidal category $\mathscr{V}$, it is convenient, as in [10, p. 49], to introduce the automorphism

$$\psi_A = (1 \otimes \varepsilon)(c_{A^*,A} \otimes 1)(c_{A,A^*} \otimes 1)(\eta \otimes 1): A \to A,$$

which is natural in those objects A with duals A^* having unit η and counit ε. The following result is stated in [10, p. 50]; the proof amounts to seeing what it means for the unit (counit) to be an arrow in the automorphism category.

Proposition 6. *Suppose $\mathscr{V}$ is any braided monoidal category. An object (A, ξ) of the automorphism category $\mathscr{V}^Z$ has a left dual if and only if the object A of $\mathscr{V}$ has a left dual A^*. Moreover, the left dual of (A, ξ) has the form (A^*, ρ^*) where the automorphism $\rho : A \to A$ is defined by the equation $\xi \rho \psi_A = 1$.*

Proposition 7. *Suppose $\mathscr{V}$ is any monoidal category. An object (A, u, ξ) of $\mathscr{Z}(\mathscr{V})^Z$ has a left dual if A has a left dual A^* in $\mathscr{V}$ and $\xi\xi\psi_{(A,u)} = 1$ where*

$$\psi_{(A,u)} = (\varepsilon \otimes 1)(u_{A^*} \otimes 1)(1 \otimes u_{A^*})(1 \otimes \eta) : A \to A.$$

Proof. Define $\varepsilon' : A \otimes A^* \to I$ to be the composite

$$A \otimes A^* \xrightarrow{\xi \otimes 1} A \otimes A^* \xrightarrow{u_{A^*}} A^* \otimes A \xrightarrow{\varepsilon} I$$

and define $\eta' : I \to A^* \otimes A$ to be the composite

$$I \xrightarrow{\eta} A \otimes A^* \xrightarrow{u_{A^*}} A^* \otimes A \xrightarrow{1 \otimes \xi} A^* \otimes A.$$

A familiar calculation [11, Section 10] using $\xi\xi\psi_{(A,u)} = 1$ shows that ε', η' are the counit and unit for A as a left dual of A^*. To be more explicit, the requirement $(\varepsilon' \otimes 1)(1 \otimes \eta') = 1$ is easily transformed to the equation $\xi\psi_{(A,u)}\xi = 1$. To prove that $(1 \otimes \varepsilon')(\eta' \otimes 1) : A^* \to A^*$ is the identity, it suffices to prove that its mate under the (ε, η) duality is the identity $A \to A$; that is, we must show that $\varepsilon(1 \otimes \varepsilon' \otimes 1)(\eta' \otimes 1 \otimes 1) = \varepsilon$. To prove this, we make use twice of the fact that the inverse of u is natural with respect to arrows between $A \otimes A^*$ and I; first, using the arrow $\varepsilon'(\xi \otimes 1) : A \otimes A^* \to I$, we obtain the rather lengthy expression

$$\varepsilon' \otimes 1 = (1 \otimes \varepsilon)(1 \otimes u_{A^*})(1 \otimes \xi^2 \otimes 1)(u_A^{-1} \otimes 1)(1 \otimes u_{A^*}^{-1})(\xi^{-1} \otimes 1 \otimes 1),$$

and second, using $\eta : I \to A \otimes A^*$, we obtain the alternative expression

$$\psi_{(A,u)} = (\varepsilon \otimes 1)(1 \otimes u_A^{-1})(u_{A^*} \otimes 1)(\eta \otimes 1) : A \to A.$$

The expression $\varepsilon(1 \otimes \varepsilon' \otimes 1)(\eta' \otimes 1 \otimes 1)$ then reduces to

$$\varepsilon u_{A^*}(\xi^2 \otimes 1)(\psi_{(A,u)} \otimes 1) u_{A^*}^{-1}$$

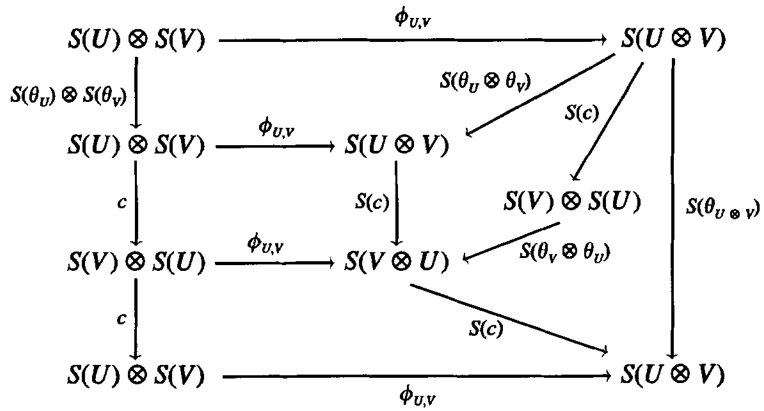
which is equal to ε as required.

The mate $X \otimes A^* \to A^* \otimes X$ of $u_X : A \otimes X \to X \otimes A$ under the duality (ε', η') gives an inverse for the mate $v : A^* \otimes X \to X \otimes A^*$ of $u_X^{-1} : X \otimes A \to A \otimes X$ under the duality (ε, η). By Proposition 3, (A^*, v) is a left dual for (A, u) in $\mathscr{Z}(\mathscr{V})$. By Proposition 6, (A^*, v, ξ^*) is a left dual for (A, u, ξ) in $\mathscr{Z}(\mathscr{V})^Z$. □

We shall now show that the automorphism category construction provides the right adjoint for the forgetful functor from balanced monoidal categories to braided monoidal categories.

Proposition 8. *Suppose $\mathscr{V}$ is any braided monoidal category and $\mathscr{C}$ is any balanced monoidal category. For all braided (strong) monoidal functors $S:\mathscr{C}\to\mathscr{V}$, there exists a unique balanced (strong) monoidal functor $T:\mathscr{C}\to\mathscr{V}^{\mathbf{Z}}$ whose composite with the forgetful functor $\mathscr{V}^{\mathbf{Z}}\to\mathscr{V}$ is S.*

Proof. Define $T(U)=(S(U),S(\theta_U))$ for each object U of $\mathscr{C}$. Since twists are natural, we obtain an arrow $T(h)=S(h)$ in $\mathscr{V}^{\mathbf{Z}}$ for each arrow $h:U\to V$ in $\mathscr{C}$. Certainly $T:\mathscr{C}\to\mathscr{V}^{\mathbf{Z}}$ is then a functor whose composite with the forgetful is S. The coherent arrows $\phi_{U,V}:S(U)\otimes S(V)\to S(U\otimes V)$ lift to arrows $\phi_{U,V}$: $T(U)\otimes T(V)\to T(U\otimes V)$ because of the following commutative diagram:



Also $I\to S(I)$ lifts to $I\to T(I)$ since $\theta_I=1$. Since the forgetful $\mathscr{V}^{\mathbf{Z}}\to\mathscr{V}$ is braided strict monoidal, it follows that T is braided (strong) monoidal because S is. Also, $T(\theta_U)=S(\theta_U)=\theta_{T(U)}$ so T is balanced. Uniqueness is clear. □

Again, there is a left biadjoint to the forgetful 2-functor taking balanced monoidal categories (and balanced strong monoidal functors) to braided monoidal categories (and braided strong monoidal functors). To be explicit, for each braided monoidal category $\mathscr{V}$ there is a balanced monoidal category $\mathscr{bal}(\mathscr{V})$ such that for all balanced monoidal categories $\mathscr{C}$ the category of balanced strong monoidal functors $\mathscr{bal}(\mathscr{V})\to\mathscr{C}$ (with monoidal natural transformations as the arrows) is (pseudo-naturally) equivalent to the category of braided strong monoidal functors $\mathscr{V}\to\mathscr{C}$ (with monoidal natural transformations as the arrows).

3. Forcing tortility

Given any balanced monoidal category $\mathscr{V}$, consider the full subcategory $\mathscr{N}(\mathscr{V})$ of $\mathscr{V}$ consisting of those objects A which have a left dual A^* satisfying the equation

$$\theta_{A^*}=(\theta_A)^*.$$

Proposition 9. *For any balanced monoidal category $\mathscr{V}$, the subcategory $\mathscr{N}(\mathscr{V})$ is closed under the balanced monoidal structure and, in fact, is tortile. If $\mathscr{C}$ is any tortile monoidal category then every balanced strong monoidal functor $S: \mathscr{C} \to V$ lands in $\mathscr{N}(\mathscr{V})$.*

Proof. The unit I is clearly in $\mathscr{N}(\mathscr{V})$. Suppose A, B are in $\mathscr{N}(\mathscr{V})$. Then we can take $(A \otimes B)^* = B^* \otimes A^*$ so that $\theta_{(A \otimes B)^*} = \theta_{B^* \otimes A^*} = c(\theta_{A^*} \otimes \theta_{B^*})c = c((\theta_A)^* \otimes (\theta_B)^*)c = (c(\theta_B \otimes \theta_A)c)^* = (\theta_{A \otimes B})^*$; so $A \otimes B$ is in $\mathscr{N}(\mathscr{V})$. Since $\mathscr{N}(\mathscr{V})$ is a full subcategory, it contains the braidings and twists, so $\mathscr{N}(\mathscr{V})$ is balanced. To see that $\mathscr{N}(\mathscr{V})$ is tortile, it remains to see that each object A of $\mathscr{N}(\mathscr{V})$ has a dual A^* which is also an object of $\mathscr{N}(\mathscr{V})$. Since $\mathscr{V}$ is balanced, $A^{**} = A$ is a left dual for A^* with counit given by the following composite:

$$A \otimes A^* \xrightarrow{\theta_A \otimes 1} A \otimes A^* \xrightarrow{c_{A,A^*}} A^* \otimes A \xrightarrow{\varepsilon} I.$$

This yields the equation $\theta_{A^{**}} = (\theta_{A^*})^*$, so A^* is in $\mathscr{N}(\mathscr{V})$.

Moving to the second statement of the proposition, we see that S preserves duals since it is strong monoidal. Then S lands in $\mathscr{N}(\mathscr{V})$ since S is balanced and $\mathscr{C}$ is tortile. □

Proposition 10. *For any balanced abelian monoidal category $\mathscr{V}$, the subcategory $\mathscr{N}(\mathscr{V})$ is closed under finite limits and colimits, and so is abelian monoidal.*

Proof. It is easy to see that $(A \oplus B)^* = A^* \oplus B^*$ and $\theta_{(A \oplus B)^*} = (\theta_{A \oplus B})^*$; so $\mathscr{N}(\mathscr{V})$ is closed under direct sums. Suppose the sequence $0 \to K \to A \xrightarrow{f} B$ is exact in $\mathscr{V}$ with A, B in $\mathscr{N}(\mathscr{V})$. Define K' to be the cokernel of $f^*: B^* \to A^*$. Then, for all X in $\mathscr{V}$, the sequence $B^* \otimes X \to A^* \otimes X \to K' \otimes X \to 0$ is exact in $\mathscr{V}$. So, for all X in $\mathscr{V}$, the sequence of abelian groups

$$0 \to \mathrm{Hom}(K' \otimes X, Y) \to \mathrm{Hom}(A^* \otimes X, Y) \to \mathrm{Hom}(B^* \otimes X, Y)$$

is exact. Comparing this with the exact sequence

$$0 \to \mathrm{Hom}(X, K \otimes Y) \to \mathrm{Hom}(X, A \otimes Y) \to \mathrm{Hom}(X, B \otimes X),$$

we see that K' is a left dual for K. So kernels of arrows between right duals are right duals. Since $\mathscr{V}$ is braided, every left dual is also a right dual. So, by a similar argument, we see that cokernels of arrows between right duals are right duals. So the objects of $\mathscr{V}$ with left duals are closed under finite limits and colimits. We have incidentally shown that taking left duals takes kernels to cokernels and cokernels to kernels. Hence, using naturality of twist, we see that, if the equation $\theta_{A^*} = (\theta_A)^*$ holds for the domain and codomain of an arrow, it holds for the kernel and cokernel of the arrow. □

Proposition 11. *Suppose B is a cobalanced bialgebra. Then there exists a cotortile Hopf algebra $D_3(B)$ unique up to isomorphism with the property that there is an equivalence of balanced monoidal categories*

$$Comod_f(D_3(B)) \xrightarrow{\sim} \mathcal{N}(Comod_f(B))$$

which commutes with the forgetful functors into Vect.

Proof. Proceed as in the proof of Proposition 2. □

4. The quantum double

The *double* $\mathcal{D}(\mathcal{V})$ of a monoidal category $\mathcal{V}$ is defined to be the tortile monoidal category $\mathcal{N}(\mathcal{Z}(\mathcal{V})^Z)$. Using Proposition 7, we see that the objects of $\mathcal{D}(\mathcal{V})$ are triples (A, u, ξ) where $\xi: (A, u) \to (A, u)$ is an automorphism in $\mathcal{Z}(\mathcal{V})$ such that A has a left dual in $\mathcal{V}$ and

$$\xi\xi(\varepsilon \otimes 1)(u_{A^*} \otimes 1)(1 \otimes u_{A^*})(1 \otimes \eta) = 1 : A \to A.$$

When $\mathcal{V}$ is left autonomous, this definition agrees with the double which is the subject of [14].

Theorem 12. *For any bialgebra B, there is a cotortile Hopf algebra $D(B)$, called the* quantum double of *B*, *unique up to isomorphism with the property that there is an equivalence of balanced monoidal categories*

$$Comod_f(D(B)) \xrightarrow{\sim} \mathcal{D}(Comod_f(B))$$

which commutes with the forgetful functors into Vect.

Proof. Take $D(B) = D_3(D_2(D_1(B)))$ and apply Propositions 2, 5 and 11. □

The original double construction of Drinfeld [*D*] applied to a finite dimensional Hopf algebra H produced a braided Hopf algebra H' with underlying vector space $H \otimes H$. Majid [17] pointed out the equivalence of braided monoidal categories

$$Mod_f(H') \xrightarrow{\sim} \mathcal{Z}\, Mod_f(H).$$

Reshetikhin–Turaev [19] showed how to construct a tortile Hopf algebra $\mathrm{RT}(H)$ (they called it a "ribbon algebra") from a finite-dimensional Hopf algebra H. Kassel–Turaev [14] showed the equivalence of tortile monoidal categories

$$Mod_f(\mathrm{RT}(H)) \xrightarrow{\sim} \mathcal{D}\, Mod_f(H).$$

Theorem 12 improves on these results by eliminating the hypothesis of finite dimensionality, producing a result for general quantum groups and their representations in the sense proclaimed in our introduction. When H is finite dimensional, a [braided, balanced] tortile structure on H^* amounts to a [cobraided, cobalanced] cotortile structure on H and there is an equivalence

$$Mod_f(H^*) \xrightarrow{\sim} Comod_f(H).$$

It follows then that $D(H^*)^*$ is isomorphic (as a tortile Hopf algebra) to $\mathrm{RT}(H)$ since we have the following equivalences of balanced monoidal categories:

$$Mod_f(D(H^*)^*) \xrightarrow{\sim} Comod_f(D(H^*)) \xrightarrow{\sim} \mathscr{D}(Comod_f(H^*)) \xrightarrow{\sim} \mathscr{D}(Mod_f(H)).$$

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